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Free convection near a thermal quadrupole

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Abstract—A conically self-similar solution of the Boussinesq equations is reported. Thermogravitational convection near a quadrupolar point singularity of a temperature field is studied. At a Prandtl number of zero the solution loses its existence when the Grashof number achieves some critical value. If the Prandtl number differs from zero, then the solution exists at any Grashof number but, when the Prandtl number tends to zero, the passage to the limit can become nontrivial. At subcritical Grashof numbers, a strong upward jet is developed. A number of problems are studied in which the flow region is bounded by a conical surface. These problems may serve as simple models of convection near a volcano, a glacier, and an iceberg.

1. INTRODUCTION

THERE ARE practically no exact solutions of Boussinesq equations. An exception is the problem of free convection between parallel planes having different temperatures [1]. In this case the equations become linear, and the thermal problem is solved separately. This example is very simplified and does not reflect the interaction between inertial and diffusive mechanisms of momentum and energy transfer which may lead to paradoxical effects.

Such effects take place in a conically similar class of Navier–Stokes equations. Under certain conditions, a vorticity is concentrated drastically near the symmetry axis and the solution loses its existence because of the appearance of a singularity [2, 3]. This phenomenon was revealed for complete Navier–Stokes equations but not for their boundary layer approximation.

As a rule, the free convection problems are solved just in the boundary layer approximation [4] and by the method of matched asymptotic expansions [5]. They comprise the problem of convection near a point heat source [6–8]. In the present work, complete Boussinesq equations are considered.

In the conically similar class, velocity components vary in inverse proportion with the distance to the coordinate origin, $u \sim 1/R$, therefore for the sake of self-similarity the buoyancy force must be proportional to R^{-3}. This dependence takes place if the temperature field has a quadrupolar point singularity. Though this statement of the problem seems rather artificial at first sight, it can serve as a simple model for a number of natural processes, such as convection near a volcano or a glacier, a cold wind from an iceberg in an ocean, and convection in a conical crater.

Due to its relative simplicity, the self-similar problem admits some analytical study. At the same time it includes such strong non-linear effects as the flow separation and collapse of vorticity and heat which are typical for conical flows of a viscous fluid.

2. REDUCTION OF THE BOUSSINESQ EQUATIONS

For the conically similar class of viscous incompressible fluid flows the velocity and pressure fields are sought in the form

$$u_R = -\nu y'(x)/R; \quad u_\theta = -\nu y(x)/(R\sin\theta)$$
$$p = p_\infty + \rho_\infty \nu^2 q(x)/R^2; \quad x = \cos\theta. \tag{1}$$

Here (r, θ, φ) are the spherical coordinates. The axisymmetric case is considered

$$u_\varphi = \partial/\partial\varphi = 0$$

where φ is the axial angle, θ is the angle between the radius-vector and the symmetry axis, the prime denotes differentiation with respect to x, and $y(x)$ is related to the Stokes stream function as $\psi = \nu R y$.

The temperature field is defined as

$$T = T_\infty + \gamma\vartheta(x)/R^3. \tag{2}$$

The substitution of equations (1) and (2) into the Boussinesq equations [9] yields the system of ordinary differential equations

$$(1-x^2)y''' - 2xy'' - (y^2)''/2 - y^2/(1-x^2) - Gr\,x\vartheta = 2q \tag{3a}$$

$$(1-x^2)q' + (1-x^2)y'' + yy' - xy^2/(1-x^2) = (1-x^2)Gr\,\vartheta \tag{3b}$$

$$(1-x^2)\vartheta'' - 2x\vartheta' + 6\vartheta = Pr(y\vartheta' + 3y'\vartheta) \tag{3c}$$

where $Gr = \beta\gamma g/\nu^2$ is the Grashof number. Eliminating q from equations (3a) and (3b) gives

$$(1-x^2)y'''' - 4xy''' - (y^2/2)''' = Gr(x\vartheta' + 3\vartheta). \tag{4}$$

It is convenient to define $F(x)$ as

$$F''' = Gr\,(x\vartheta' + 3\vartheta); \quad F(1) = F'(1) = F''(1) = 0. \tag{5}$$

NOMENCLATURE

$A, a, b, b_1, b_2, b_3, C$ numerical constants
F, f, f_0, f_1 auxiliary functions
g acceleration of gravity
Gr, Gr_* Grashof number and its critical value
Gr_Q Grashof number based on heat flux
$Gr_l, Gr_u, Gr_l^-, Gr_u^-$ lower and upper estimates of Gr_* and $-Gr_*$
p pressure
Pr Prandtl number
Q heat flux
q dimensionless pressure
R spherical radial coordinate
r cylindrical radial coordinate
T temperature
u_R, u_θ, u_φ spherical components of velocity
u_r radial cylindrical velocity
U, w auxiliary functions
x auxiliary argument
x_p, x_s, x_w, x_1 positions of pole $y(x)$, separatrix, wall, of the zero of $f(x)$
y dimensionless stream function
y_l solution of linearized problem
y_i, y_o main terms of inner and outer expansions
z vertical axis (axis of symmetry).

Greek symbols
α scale factor
β coefficient of thermal expansion
γ coefficient in the function $T(R)$
δ scale factor
$\varepsilon, \varepsilon_1$ small parameters
η boundary layer coordinate
θ spherical coordinate, angle between the symmetry axis and radius vector
ϑ dimensionless temperature
ϑ_0 regular solution at $Pr = 0$
λ thermal conductivity
ν kinematic viscosity
ρ density
τ shear stress
φ spherical coordinate, longitude
ψ Stokes stream function.

Subscripts
∞ parameters of ambient medium
s separation parameters.

Superscript
$'$ differentiation.

Then equation (4) can be integrated three times to give

$$(1-x^2)y'+2xy-y^2/2 = F(x)-C(1-x)^2 \quad (6)$$

where the integration constant C has to be found from boundary conditions. While integrating (4) the requirements of solution regularity at $x = 1$, $y(1) = 0$, $|y'(\infty)| < \infty$ are used.

3. THE CASE $Pr = 0$

At the Prandtl number equal to zero, the energy equation is separated from the system. Equation (3c) yields

$$(1-x^2)\vartheta''-2x\vartheta'+6\vartheta = 0. \quad (7)$$

The regular solution of equation (7) is

$$\vartheta = \vartheta_0(x) = 3x^2-1. \quad (8)$$

The function $\vartheta_0(x)$ changes its sign in the interval $-1 \leqslant x \leqslant 1$. When $Gr > 0$, then the temperature in the cone $|x| > 1/\sqrt{3}$ is greater than the ambient temperature; when $Gr < 0$, the opposite is true. Substituting equation (8) into equation (5) and integrating one finds

$$F = Gr\,x(1-x^2)^2/4. \quad (9)$$

Suppose the flow region is bounded by a conical surface, i.e. $x \geqslant x_w$, and the adherence conditions are to be satisfied $y(x_w) = y'(x_w) = 0$. Then according to equations (6) and (9), $C = 1/4\ Gr\ x_w(1+x_w)^2$, and equation (6) may be rewritten as

$$(1-x^2)y'+2xy-y^2/2 = Gr/4(1-x)^2f(x)$$

$$f(x) = x(1+x)^2-x_w(1+x_w)^2. \quad (10)$$

When $|Gr| \ll 1$, the last term on the left-hand side of equation (10) may be neglected, and then the solution is found explicitly

$$y = y_L(x) = Gr/8(1-x)(1+x+2x_w)(x-x_w)^2. \quad (11)$$

When $x_w \geqslant -1/3$, the flow has a one-cellular structure. It is ascending for $Gr > 0$ and descending for $Gr < 0$. With $x_w < -1/3$ the flow has a two-cellular structure. The interface surface equation is

$$x = x_s = -1-2x_w.$$

At $x_w = -1/3$, $x_s = -1/3$, when $x_w \to -1$ and $x_s \to 1$, this means that with a decrease of the cone angle, the reverse flow (separation) first originates only near the cone surface; thereafter, the recirculation region becomes wider, and at $x_w = -1$ (a quadrupole in an infinite medium) the flow is descending throughout for $Gr > 0$ and ascending for $Gr < 0$.

If the Grashof number is not small, the separation occurs as before when x_w, decreasing, passes the value $-1/3$. For this to be verified, use will be made of the fact that the wall friction becomes zero at the parameters corresponding to separation. The shear stress is proportional to $y''(x_w)$

$$\tau_w = -y''(x_w)\rho v^2/R^2.$$

Differentiating equation (10) once and substituting $x = x_w$, with regard for $y(x_w) = f(x_w) = 0$, yields

$$y''(x_w) = Gr/4(1+x_w)(1+3x_w)$$

from which it is seen that $y''(x_w)$ changes its sign at $x_w = -1/3$.

The analysis of the solution properties in the case of $|Gr| \gtrsim 1$ is alleviated by the substitution of [3]

$$y = -2(1-x^2)U'/U \tag{12}$$

into equation (10) which is thus reduced to a linear relationship

$$U'' + \frac{Gr\, f(x)}{8(1+x)^2}U = 0 \tag{13}$$

$$U(x_w) = 1; \quad U'(x_w) = 0. \tag{14}$$

The condition $U'(x_w) = 0$ follows from $y(x_w) = 0$ and from equation (12). Normalization, $U(x_w) = 1$, is made without loss of generality. Thus, there is an initial value problem for $U(x)$. This has a single solution according to the well-known theorems.

For the existence of the regular solution y, it is necessary and sufficient that U would not turn to zero in $[x_w, 1]$, since the zero value of $U(x)$ is the pole of $y(x)$. At $Gr = 0$, $U \equiv 1$. Therefore, when $|Gr| \ll 1$, the curvature $|U''(x)| \ll 1$ and therefore the first zero of $U(x)$ (if it exists), x_p, is placed far from the interval, $x_p \gg 1$. However, with an increase of $|Gr|$, the value of x_p decreases and may pass 1.

First, the case $x_w \geqslant -1/3$ will be considered, then $f(x) \geqslant 0$ when $x \geqslant x_w$. This yields $U''(x) \geqslant 0$, $U'(x) \geqslant 0$, $U(x) \geqslant 1$ when $Gr < 0$, which means that the function $U(x)$ has no zeroes and the solution $y(x)$ exists for any $Gr < 0$. In this case $y(x) < 0$ and the flow is downward.

The situation is different when $Gr > 0$. Problems (13) and (14) may be transformed to the integral equation

$$U(x) = 1 - \frac{Gr}{8}\int_{x_w}^{x} \frac{x-t}{(1+x)^2} f(x)U(x)\,dx. \tag{15}$$

Since the function $U(x)$ decreases in the interval $[x_w, 1]$ and $U''(x) < 0$, it is located within the following limits.

Using these estimates and the fact that $U(1) = 0$ at $Gr = Gr_*$, it is possible to find from equation (15) that

$$Gr_l \leqslant Gr_* \leqslant Gr_u$$

where

$$Gr_l = 8\left\{\frac{1-x_w^3}{6} + b_1 \ln\frac{2}{1+x_w} - x_w(1-x_w)\left(1+\frac{3x_w}{2}\right)\right\}^{-1}$$

$$Gr_u = 8(1-x_w)\left\{\frac{1}{12} + 2b_1(\ln 4 - 1) - b_2 - b_3\right\}^{-1}$$

$$b_1 = x_w(1+x_w)^2$$

$$b_2 = \frac{x_w^3(3-2x_w)}{6} + b_1\left[\ln(1+x_w) + \frac{2}{1+x_w}\right]$$

$$b_3 = \frac{x_w^3}{6} - \frac{x_w^4}{12} + b_1[(3+x_w)\ln(1+x_w)1 - 1 - x_w] - b_2 x_w.$$

When $Gr < Gr_l$, the regular solution $y(x)$ is known to exist, and when $Gr > Gr_u$ it is known to be absent.

If $x_w < -1/3$, then the estimates are more sophisticated because $f(x)$ alternates in sign. An exact value of Gr_* can be found numerically by integrating (13) and by selecting Gr_* so that $U(1) = 0$. The results of calculation with the analytical estimates are presented in Fig. 1. When $x_w \to -1$, the function $Gr_*(x_w)$ has an asymptotic form $Gr_* = 31.4/(1-x_w)^3$.

At $x_w = 0$ problem (13) is simplified to

$$U'' + \frac{Gr\,x}{8}U = 0; \quad U(0) = 1; \quad U'(0) = 0.$$

Its solution is the Airy function [10] and can be presented in the form of a regular series

$$U = \sum_{n=0}^{\infty} a_n\left(-\frac{Gr\,x^3}{48}\right)^n; \quad a_0 = 1$$

$$a_n = 2a_{n-1}/[n(3n-1)]; \quad n = 1, 2, \ldots.$$

The value $U(1)$ turns to zero at $Gr = Gr_* = 62.7$.

Now we will find what physical phenomena are associated with the appearance of singularity. For x

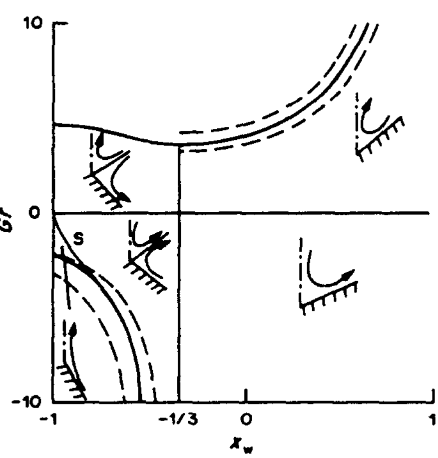

FIG. 1. Map of convection regimes at $Pr = 0$.

close to x_p, one may write

$$U(x) = U'(x_p)(x-x_p)+1/6U'''(x_p)(x-x_p)^3+\cdots$$

whence, due to equation (12)

$$y(x) = 2\frac{1-x^2}{x_p-x}\left[1+\frac{U'''(x_p)}{3U'(x_p)}(x-x_p)^2+\cdots\right].$$

At $x_p = 1$ the following equation is obtained:

$$y = y_0(x) = 2(1+x)+O[(1-x)^2].$$

Thus, $y(x)$ remains bounded at critical parameters, but $y_0(1) = 4$, i.e. the non-permeability condition is broken down and a fluid sink is formed at the axis. Since in the subcritical case $y(1) = 0$, then the derivative $y'(1) \to -\infty$ when $Gr \to Gr_*$. Now, introducing the small parameter $\varepsilon = -1/y'(1)$ and the new variable $\eta = (1-x)/\varepsilon$, and after substitution of the argument in equation (10) and the limiting transition $\varepsilon \to 0$, the following equation can be obtained for the axial boundary layer:

$$\eta y_i' = y_i - y_i^2/4; \quad y_i(0) = 0$$

where y_i is the main part of the inner asymptotic expansion of y.

The solution $y_i(\eta) = 4\eta/(4+\eta)$ coincides with the well-known Schlichting solution for a round submerged jet [11].

Thus, at subcritical parameters a strong upward jet is formed near the axis. At $Gr = Gr_*$ it collapses when the axial velocity and the jet momentum become infinite, but the external velocity field remains finite and corresponds to the flow induced by the sink at the axis. Mathematically it is related to the leading term of the outer asymptotic expansion in terms of $\varepsilon: y = y_0(x)+\varepsilon y_{01}(x)+\cdots$.

In the range $-1 \leqslant x_w \leqslant -1/3$ the collapse is also possible when $Gr < 0$. This may be found analytically. The function $f(x)$ is negative in $[x_w, x_1)$ and positive in $(x_1, 1]$ where

$$x_1 = 1/2[-x_w(4+3x_w)]^{1/2}-1-x_w/2.$$

Therefore, for the near-collapse situation $U(x)$ has the following bounds: $U \leqslant 1$ for $x_w \leqslant x \leqslant 1$ and $U \geqslant (x_1-x)/(x_1-x_w)$ for $x_w \leqslant x \leqslant x_1$.

Let

$$f_0(t) = f(t)(x_1-t)/(1+t)^2$$

and

$$f_1(t) = f(t)(1-t)/(1+t)^2.$$

The lower estimate will be found for the quantity $|Gr_*|$. One has

$$0 = U(1)-|Gr_*|/8\int_{x_w}^{x_1}|f_1|U\,dt + |Gr_*|/8\int_{x_1}^{1} f_1U\,dt \geqslant 1-|Gr_*|/8\int_{x_w}^{x_1}|f_1|\,dt.$$

While forming the inequality, the function U in the first integral is replaced by 1 and the second integral is discarded. Hence, $|Gr_*| \geqslant Gr_l^-$ where

$$Gr_l^- = 8\left\{\frac{x_w^2-x_1^2}{2}-\frac{x_w^3-x_1^3}{3} + b_1\ln\frac{1+x_w}{1-x_1}+2x_w(x_1-x_w)(1+x_w)/(1+x_1)\right\}^{-1}.$$

The upper estimate will be

$$0 < U(x_1) = 1-Gr_*/8\int_{x_w}^{x_1} f_0U\,dt \leqslant 1-\frac{|Gr_*|}{8}\int_{x_w}^{x_1}\frac{x_1-t}{x_1-x_w}|f_0|\,dt.$$

Hence, $|Gr_*| \leqslant Gr_u^-$, where

$$Gr_u^- = 96b/(1+x_w)^2\{12x_w[b^2+2b - 2(1+b)\ln(1+b)]-3(x_w+x_1)b^3\}^{-1};$$

$$b = (x_1-x_w)/(1+x_w).$$

Particularly, at $x_w = -1$, $Gr_l^- = 9.6 < |Gr_*| < 32 = Gr_u^-$; $Gr_* \approx -11$ and when $x_w \to -1/3$, one may see that $|Gr_*| \to \infty$ and, asymptotically, $2 < |Gr_*\varepsilon_1^3| < 96$, where $\varepsilon_1 = -1/3-x_w$ $(Gr_* \approx -27\varepsilon_1^3)$.

The results of numerical calculations (solid line) and analytical estimates (dashed lines) are shown in Fig. 1. At the critical parameters, $U \geqslant 0$ and $U' \leqslant 0$, and the flow is upward throughout. These regimes also persist in the neighbourhood of the boundary curve for the solution's existence in the parameter plane. The upper boundary of this region is shown by a thin curve S in Fig. 1. For this boundary $y'(1) = 0$. When the curve S passes from left to right, a recirculation flow region appears near the axis with downward movement. There is a conical surface which separates the two cells of the flow. The angle of the cone increases with x_w and at $x_w = -1/3$ the cone touches the wall. When $x_w > -1/3$, the convection has a one-cellular structure.

In the range $-1/3 \leqslant x_w < 1$, the value of $-Gr$ may be arbitrarily large. Then $|y|$ becomes large and the linear terms on the left-hand side of equation (10) may be neglected in the flow core, i.e. $y \approx y_0(x)$ where

$$y_0(x) = -|Gr/2|^{1/2}(1-x)[x(1+x)^2-x_w(1+x_w)^2]^{1/2}.$$

At $x = 1$ the function $y_0(x)$ turns to zero and has a bounded derivative. At the wall the function $y_0(x)$ is also zero, but $|y_0'(x)| \to \infty$ when $x \to x_w$. Consequently, there is a boundary layer near the wall. To form the boundary layer equations, new variables have to be introduced

$$x = x_w+\delta\eta; \quad y = -\varkappa w$$

$$\delta = \left[\frac{4(1+x_w)}{Gr(1+3x_w)}\right]^{1/3}; \quad \varkappa = \frac{1-x_w^2}{\delta}.$$

Substituting the above variables in equation (10) and letting $|Gr| \to \infty$, it is possible to obtain

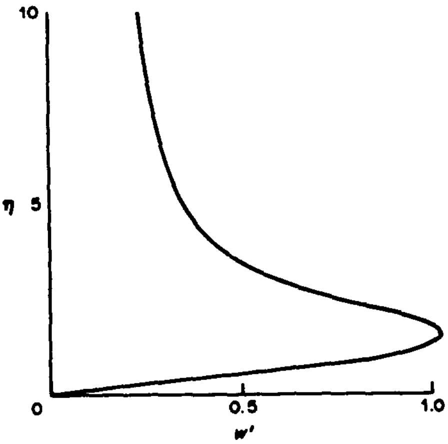

FIG. 2. Asymptotic velocity distribution for a near-wall jet.

$$dw/d\eta = w' = \eta - w^2/2; \quad w(0) = 0.$$

The solution of this equation can be expressed explicitly in terms of the Airy function. The quantity w' reaches its maximum value of 1.03 at $\eta = 1.5$ (Fig. 2). Thus the near-wall jet is developed with the maximum velocity $\sim |Gr|^{2/3}$ and with $\sim |Gr|^{-1/3}$.

4. THE QUADRUPOLE ON THE PLANE

Now, the problem will be considered at $Pr \neq 0$ for the upper half-space, $0 \leqslant x \leqslant 1$. Because of the quadrupolar type of singularity, the values of the functions $\vartheta(1)$ and $\vartheta(0)$ have opposite signs. When $\vartheta(1) > 0$ the problem may serve as the simplest model of heat convection near a volcano, and with $\vartheta(1) < 0$ as that near a glacier.

From equation (5) it follows that

$$2F(0) = \int_0^1 x^2 F''' \, dx = Gr \int_0^1 (x^3 \vartheta)' \, dx = Gr\,\vartheta(1).$$

Substituting $x = 0$ and $y = y'$ into equation (6) yields $C = F(0) = Gr\,\vartheta(1)/2$. The selection of some tentative value of $y'(1)$ gives an initial-value problem for the system of equations (3c), (5), and (6). This has to be integrated from $x = 1$ and 0. The regularity requirements at $x = 1$ yield

$$\vartheta'(1) = 3\vartheta(1)[1 - Pr\,y'(1)];$$

$$\vartheta''(1) = \vartheta'(1)[1 - Pr\,y'(1)] - 3/4Pr\,\vartheta(1)y''(1);$$

$$y''(1) = y'(1)[1 - y'(1)/2] - Gr\,\vartheta(1)/2.$$

The quantity $y'(1)$ must be selected (by shooting) in such a way that after the integration it can satisfy $y(0) = 0$. Then, the renormalization of ϑ and Gr was made so that the function $\vartheta(0)$ could be equal to -1. The regime map is shown in Fig. 3. When $Gr < 0$, the flow is downward and when $|Gr| \gg 1$ a near-wall jet is developed. Unlike the case $Pr = 0$, the thermal boundary layer also appears in addition to the dynamic layer (Fig. 4; $r = R \sin\theta$, $z = R\cos\theta$, $u_r = -v/r[(1-x^2)y' + xy]$ is the horizontal velocity).

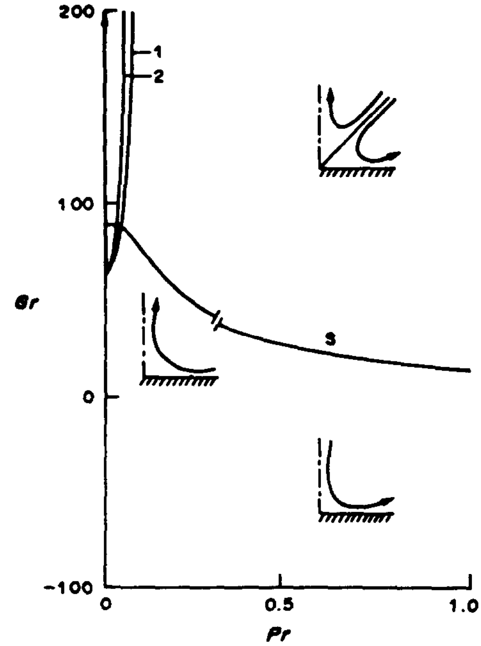

FIG. 3. Map of convection regimes at $x_w = 0$.

When $Gr > 0$, there is a region on the plane $\{Pr, Gr\}$ which corresponds to upward convective motion. This region is bounded by curve S in Fig. 3. As Gr increases, separation and flow reversal occurs near the wall. The separation is due to the unfavourable effect of the pressure gradient. The pressure distribution over the wall is governed by the value $y'''(0)$, as follows from equations (1) and (3a) and the boundary conditions:

$$p = p_\infty + y'''(0)\rho v^2/(2R^2).$$

A simple, but rather cumbersome, analysis shows that $y'''(0) = 3/32Pr\,Gr^2 + O(Pr\,Gr^2)$ near the axes $Pr = 0$ and $Gr = 0$. Consequently, the pressure increases as the symmetry axis is approached and makes the convergent motion near the plane difficult. When $Pr = 0$, the separation happens at $Gr = Gr_s = 88$. With an increase of Pr, Gr_s decreases and, asymptotically, $Pr\,Gr_s = Ra_s \approx 16$. If Pr is fixed and Gr increases, the angle of the separating cone (see the sketch above curve S in Fig. 3) decreases and except for a narrow region near the axis, the flow is downward just as for $Gr = 0$. Physically this means that the heat of a volcano is entrained by an upward jet and a cold wind blows near the surface just as in the case of a glacier.

5. THE NON-TRIVIAL LIMIT IN THE CASE OF $Pr \to 0$

If $Pr \neq 0$, the solution exists at any value of Gr. Curves 1 and 2 in Fig. 3 relate to fixed values of $y'(1)$

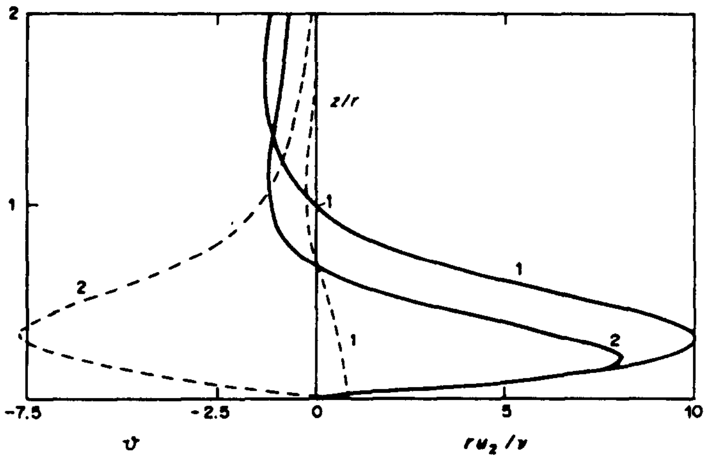

FIG. 4. Vertical distribution of temperature (– –) and of horizontal velocity (——). 1, $Pr = 0$, $Gr = -200$; 2, $Pr = 0.7$, $Gr = -35$.

(1, $y'(1) = -460.5$; 2, $y'(1) = -3000$). The curves $y'(1) = \text{const.}$ tend to the half-axis $Pr = 0$, $Gr \geqslant Gr_* = 62.7$ when $y'(1) \to -\infty$. If Gr is fixed and $Pr \to 0$, then with $Gr < Gr_*$ the limiting distributions of velocity and temperature coincide with the distributions at $Pr = 0$.

However, the situation is different when $Gr > Gr_*$. The evolution of $y(x)$ and $\vartheta(x)$ is shown in Figs. 5 and 6. Curves 1 and 2 correspond to the intersection of curves 1 and 2 in Fig. 3 with the line $Gr = 150$. As Pr decreases, a jet is formed near the axis just as in the case with $Pr = 0$ and $Gr \to Gr_*$, but when $Pr = 0$, $y > 0$ and the flow is upward. In the case of $Gr = 150$ and $Pr \to 0$, the limiting function $y(x)$ changes its sign in the interval $(x_w, 1)$ (curve 0_+ in Fig. 5). A more interesting metamorphosis takes place with $\vartheta(x)$. Its distribution for $Pr \ll 1$ differs drastically from that for $Pr = 0$ (curve 0 in Fig. 6). The calculations show that as Pr decreases, $\vartheta(1)$ grows without bound.

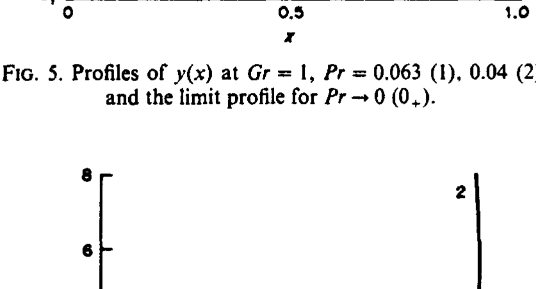

FIG. 5. Profiles of $y(x)$ at $Gr = 1$, $Pr = 0.063$ (1), 0.04 (2) and the limit profile for $Pr \to 0$ (0_+).

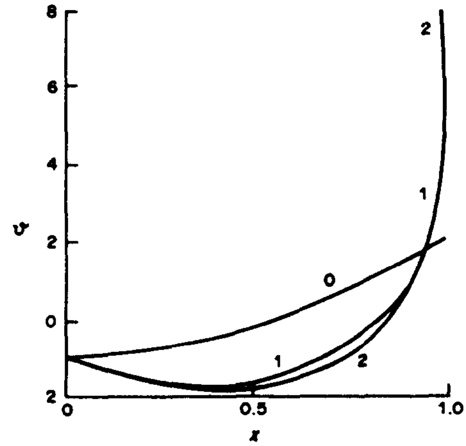

FIG. 6. Profiles of temperature. Notation is the same, but the curve 0 corresponds to the regular solution at $Pr = 0$.

To understand the reason for this behaviour, consider the energy equation (3c). If $x \neq 1$ is fixed, the convective terms tend to zero when $Pr \to 0$. But with $Pr \to 0$, $y'(1) \to -\infty$ and the limiting $y'(x)$ acquires singularity of the Dirac function type with the factor 4. Therefore when $Pr \to 0$, the convective heat transfer remains predominant although in the gradually reducing axial region. In the limit, a heat source is formed at the axis.

To take into account this phenomenon, referring to equation (7), which relates to the case $Pr = 0$, and considering not the regular but the common solution, which is

$$\vartheta = 3x^2 - 1 + A\left[(3x^2-1)\ln\frac{1+x}{1-x} - 6x\right] \quad (16)$$

where the normalization $\vartheta(0) = -1$ is used, the solution has a logarithmic singularity at $x = 1$. Physically this means that there is a heat source at the axis. The coefficient A can be found as follows. Substituting equation (16) into equation (5) and integrating gives

$$F(x) = \frac{Gr}{4}(1-x^2)^2\left[x - A\left(2 - x\ln\frac{1+x}{1-x}\right)\right].$$

Due to the singularity of $\vartheta(x)$, the condition $F'''(1) = 0$ is conserved only for the analytical part of $F(x)$ and a symmetry condition is used for the singular part. Then taking into account that $C = F(0) = -A\,Gr/4$, we obtain from equation (6) the equation for

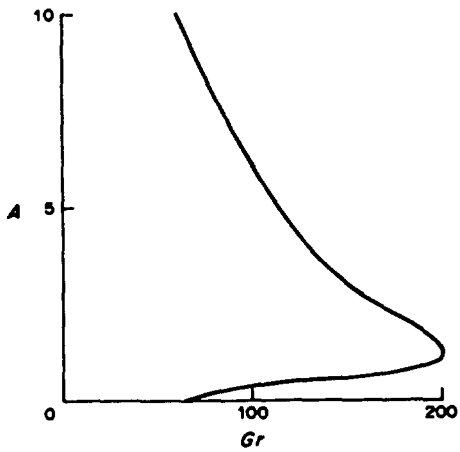

FIG. 7. Dependence of the coefficient of the logarithmic singularity in equation (16) on the Grashof number at $Pr = 0$.

the main part $y_o(x)$ of the outer asymptotic expansion $y = y_o(x)+\varepsilon y_1(x)+\cdots$:

$$(1-x^2)y_o'+2xy_o-y_o^2/2 = Gr\,x(1-x^2)^2/4$$
$$+A\,Gr\left[\frac{x(1-x^2)^2}{4}\ln\frac{1+x}{1-x}-\frac{(1-x^2)^2}{2}+\frac{(1-x)^2}{2}\right]. \quad (17)$$

This equation has to be integrated with the conditions $y_o(1) = 4$; $y_o(0) = 0$. The differentiation of equation (17) and substitution of $x = 1$ yields $y_o'(1) = 2$. Then equation (17) can be integrated from $x = 1$ as an initial-value problem, and A can be selected (by shooting) so as to satisfy $y_o(0) = 0$. The function $Gr(A)$ is found numerically and is shown in Fig. 7. The solutions, corresponding to the lower branch of the curve, are limits for the solutions with $Pr \to 0$. In Fig. 6 the distribution of $\vartheta(x)$ for $Pr = 0.04$ is graphically indiscernible with the solution of equations (16) and (17) at $Gr = 150$.

Differentiating (17) and taking $x = 0$ one finds that

$$y_o''(0) = Gr(1/4-A); \quad y_o''' = y_o'''(0) = 5A\,Gr.$$

Hence, the unfavourable pressure gradient increases with A. At $A = 1/4$ the value $y_o''(0)$ changes its sign and separation occurs.

It is seen from Fig. 7 that if Gr is sufficiently large ($Gr > 202$), solutions with logarithmic singularity are also absent. Numerical analysis indicates that in this case the singularity at $x = 1$ becomes more sharp than the logarithmic one in the limit $Pr \to 0$.

The curve in Fig. 7 bounds on plane $\{Gr, A\}$ the region of solution existence for another problem when, besides the quadrupole, there is a linear source of heat at the axis and A is a factor of the heat flux from the axis. Solutions which satisfy $y(1) = 0$, $|y'(1)| < \infty$ exist in the region located to the left of the curve.

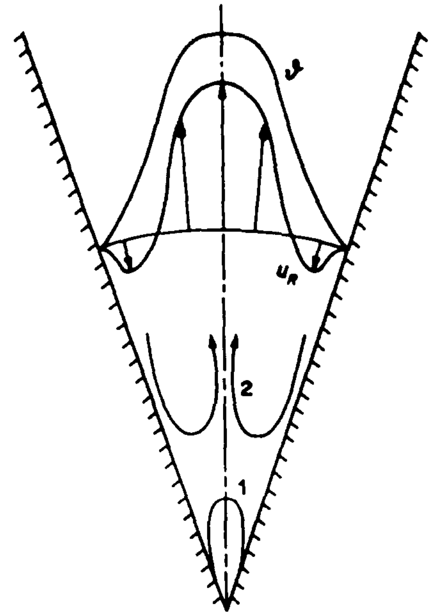

FIG. 8. Free convection in a cone with an isothermal wall induced by a heat source at the vortex. The isotherm (1), the stream line (2) and the angular distributions of radial velocity u_r and temperature ϑ are typical.

6. CONVECTION WITH THE SIGN-DEFINITE TEMPERATURE FIELD

A certain physical interest attaches to the problem when the conical wall has a uniform temperature and there is a heat source at the coordinate origin. If $Pr = 0$ or $|Gr| \ll 1$, such a situation is realized inside a cone with the angle corresponding to $x_w = 1/\sqrt{3} \approx 0.58$. In this case the source strength may be characterized by the quantity QR^2 where Q is the heat flux through a part of the spherical surface of radius R bounded by the cone $x = x_w$. The QR^2 value is independent of R. A decrease in the heat flux with the growth of R is caused by heat loss through the side wall of the cone.

Let the Prandtl number be fixed and equal to $Pr = 0.7$. For any Gr one may find such x_w that $\vartheta(x_w)$ could be equal to zero and $\vartheta(x)$ could have a uniform sign in $(x_w, 1]$. The value of x_w tends to 1 when $Gr \to \infty$, and $x_w \to 0$ when $Gr \to -\infty$. With $x_w \ll 1$ the flow structure is the same as in Fig. 4 (curve 2) with the difference that the temperature turns to zero at the wall. This problem seems to model air convection near an iceberg in the sea.

For the opposite case $x_w \gg 1$, the pattern of convection in a conical crater $0.9442 = x_w \leqslant x \leqslant 1$ at $Gr_Q = \beta g Q R^2/(\lambda \nu^2) = 29\,651$ is shown in Fig. 8.

Obviously, the stringent dependence $x_w(Gr)$ imposed by the self-similarity restricts the applications of this problem. However, a viscosity ν may be assumed as an eddy using the Boussinesq turbulence model. Then, taking into account the tendency of turbulent flows (particularly jet-like) to be self-similar, it is possible to assume that the turbulent Grashof number (i.e. turbulent viscosity) is such that the self-similarity can be fulfilled. In this sense the above problem may serve as a model of turbulent free convection in a conical crater.

7. CONCLUSIONS

In order to find self-similar conical solutions of the complete Boussinesq equations, the point quadrupolar singularity of a temperature field is considered. For such flows, molecular and convective heat and momentum transfer can be of the same order of magnitude over the entire flow region, even for the case when the velocity is arbitrarily high. This property leads to some unusually interesting mathematical and physical consequences.

(1) The solution may lose its existence when the Grashof number attains a certain finite critical value. This crisis occurs at $Pr = 0$ and it is preceded by the appearance of strong upward jets.

(2) Variation in the geometry of the region and in the change of the quadrupole may cause flow separation from the wall and the origination of two-cellular convection patterns.

(3) In problems of convection near both a volcano and a glacier (i.e. irrespective of the sign of the Grashof number), a strong cold fan near-wall jet is formed.

(4) The limit of solution when $Pr \to 0$ can differ from the solution at $Pr = 0$.

(5) In spite of the quadrupolar singularity, the problem can be formulated so that the temperature is greater or smaller throughout than that in the ambient medium. For any value of the conical crater angle, a quite definite Grashof number is found which can be treated as an invariant of the turbulent regime.

Though the solutions reported are physically interpreted where possible, they may serve only as very simplified models of the natural processes mentioned.

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CONVECTION NATURELLE PRES D'UN QUADRIPOLE THERMIQUE

Résumé—On considère une solution coniquement affine de l'équation de Boussinesq en étudiant la convection thermogravitationnelle près d'une singularité quadripolaire d'un champ de température. A un nombre de Prandtl nul, la solution perd son existence lorsque le nombre de Grashof atteint quelques valeurs critiques. Si le nombre de Prandtl diffère de zéro, la solution existe pour un nombre de Grashof quelconque mais lorsque le nombre de Prandtl tend vers zéro, le passage à la limite peut ne pas être trivial. A des nombres de Grashof subcritiques se développe un fort jet ascendant. On étudie un certain nombre de problèmes dans lesquels la région d'écoulement est limitée par une surface conique. Ces problèmes peuvent servir de modèles simples de convection près d'un volcan, d'un glacier et d'un iceberg.

FREIE KONVEKTION IN DER UMGEBUNG VON VIER PUNKTFÖRMIGEN WÄRMEQUELLEN

Zusammenfassung—Eine konische, selbst ähnliche Lösung der Boussinesq-Gleichung wird vorgestellt. Die temperaturbedingte natürliche Konvektion in der Umgebung von vier singulären Punkten in einem Temperaturfeld wird untersucht. Für die Prandtl-Zahl null existiert bei einem bestimmten kritischen Wert der Grashof-Zahl keine Lösung. Wenn die Prandtl-Zahl größer als null ist, existiert für alle Grashof-Zahlen eine Lösung; nur wenn die Prandtl-Zahl gegen null strebt, kann im Grenzbereich die Lösung mehrdeutig werden. Wenn die Grashof-Zahl unterhalb des kritischen Wertes bleibt, bildet sich eine starke Auftriebsströmung aus. Eine Reihe von Fällen wird untersucht, in denen die Strömung durch eine konische Fläche eingegrenzt ist. Diese Fälle können als einfache Abbildung der Konvektion in der Umgebung von Vulkanen, Gletschern oder Eisbergen dienen.

СОВБОДНАЯ КОНВЕКЦИЯ ВБЛИЗИ ТЕПЛОВОГО КВАДРУПОЛЯ

Аннотация—Получено автомодельное конически симметричное решение уравнений тепловой конвекции в приближении Буссинеска. Движение вызывается точечной особенностью поля температуры типа квадруполя. При числе Прандтля, равном нулю, решение теряет существование, когда число Грасгофа достигает критического значения. Если число Прандтля не равно нулю, то решение существует при всех числах Грасгофа, но предельный переход, когда число Прандтля стремится к нулю, может носить нетривиальный характер. В околокритической ситуации формируется сильная восходящая струя. Рассмотрен ряд задач, в которых область течения ограничена конической поверхностью. Они могут служить простейшими моделями конвекции вблизи вулкана, ледника или айсберга.